MATH 105 101 Midterm 2 Sample 5 Solutions

1. (20 marks)

(a) (5 marks) Determine the intervals on which the following function is concave up or concave down:

$$F(x) = \int_0^x te^t \, dt.$$

Solution: In order to find the intervals on which the function is concave up or concave down, we want to find inflection points. So, first we need the first derivative using the Fundamental Theorem of Calculus Part I, and the second derivative:

$$\frac{dF}{dx} = xe^x,$$
$$\frac{d^2F}{dx^2} = e^x(1+x).$$

The second derivative is zero only for $x = -1$. Testing the sign of the second derivative on each subintervals, we get:

- On $(-\infty, -1)$: $\frac{d^2F}{dx^2} < 0$ (for example, $\frac{d^2F}{dx^2}\mid_{x=-2} = -e^{-2}$). So, the function is concave down.
- On $(-1, \infty)$: $\frac{d^2F}{dx^2} > 0$ (for example, $\frac{d^2F}{dx^2}\mid_{x=0} = 1$). So, the function is concave up.

(b) (5 marks)

(c) (5 marks) Find the definite integral:

$$\int_{-\pi/2}^{\pi/2} \sin^7(x) \, dx.$$

Solution: First, note that $f(x) = \sin^7(x)$ is an odd function, because:

$$f(-x) = \sin^7(-x) = (-\sin(x))^7 = -\sin^7(x) = -f(x).$$

Then, since for odd functions, $\int_{-a}^{a} f(x)\, dx = 0$, we get that $\int_{-\pi/2}^{\pi/2} \sin^7(x)\, dx = 0$.
Remark: *You can also solve this by substitution with $u = \cos(x)$ and $du = \sin(x)\, dx$, but it will be long and somewhat tedious. We can just make use of symmetry when we can to save time.*

(d) (5 marks) Compute the Right Riemann sum for $f(x) = \sin^2(x)$ on the interval $[0, \pi]$ using $n = 6$ equal subintervals. **Simplify the answer.**

Solution: We have $a = 0$, $b = \pi$, and $n = 6$. So, $\Delta x = \frac{b-a}{n} = \pi/6$. For Right Riemann sum, we have:

$$x_k^* = a + k\Delta x = \frac{k\pi}{6}.$$

Thus, The Right Riemann sum for $f(x) = \sin^2(x)$ on $[0, \pi]$ using $n = 6$ equal subintervals is:

$$\begin{aligned}
&\Delta x(f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*) + f(x_5^*) + f(x_6^*)) \\
&= \frac{\pi}{6}\left(\sin^2\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{3}\right) + \sin^2\left(\frac{\pi}{2}\right) + \sin^2\left(\frac{2\pi}{3}\right) + \sin^2\left(5\frac{\pi}{6}\right) + \sin^2(\pi)\right) \\
&= \frac{\pi}{6}\left(\frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} + 0\right) = \frac{\pi}{2}.
\end{aligned}$$

(e) (5 marks) Find the definite integral

$$\int_{-\infty}^{0} e^{3x}\, dx.$$

Solution: This is an improper integral on unbounded interval, and

$$\int_{-\infty}^{0} e^{3x}\, dx = \lim_{a\to-\infty} \int_{a}^{0} e^{3x}\, dx.$$

Use substitution with $u = 3x$, we get $du = 3\, dx$. So,

$$\int e^{3x}\, dx = \int \frac{e^u}{3}\, du = \frac{e^u}{3} + C = \frac{e^{3x}}{3} + C.$$

So,

$$\lim_{a\to-\infty} \int_{a}^{0} e^{3x}\, dx = \lim_{a\to-\infty} \frac{e^{3x}}{3} \,|_a^0 = \lim_{a\to-\infty} \frac{1}{3} - \frac{e^{3a}}{3} = \frac{1}{3},$$

since as $a \to -\infty$, then $3a \to -\infty$, and $e^{3a} \to 0^+$. Thus, the limit exists, and the improper integral converges to $\frac{1}{3}$.

2. (10 marks) Let $f(x) = x^3 - \cos(x)$ on $[-\pi, \pi]$.

(a) (5 marks) Compute an approximation of $\int_{-\pi}^{\pi} f(x)\,dx$ using Simpson's Rule with $n = 4$ equal subintervals. **Simplify the answer.**

Solution: So, $a = -\pi$, $b = \pi$, $f(x) = x^3 - \cos(x)$, $n = 4$ and $\Delta x = \frac{b-a}{n} = \pi/2$. Using $x_k = a + k\Delta x$, the five grid points are:

$$x_0 = -\pi, \quad x_1 = -\pi/2, \quad x_2 = 0, \quad x_3 = \pi/2, \quad x_4 = \pi.$$

Then,

$$\begin{aligned} S_4 &= \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)) \\ &= \frac{\pi}{6}\left((-\pi^3 - \cos(-\pi)) + 4(-\frac{\pi^3}{8} - \cos\frac{-\pi}{2}) + 2(0 - \cos 0) + 4(\frac{\pi^3}{8} - \cos\frac{\pi}{2}) + (\pi^3 - \cos\pi)\right) \\ &= \frac{\pi}{6}(1 + 2 - 1) = \frac{\pi}{3}. \end{aligned}$$

(b) (5 marks) Find an upper bound of the error for the approximation in part (a).

Solution: First, we find the derivatives:

$$f'(x) = 3x^2 + \sin(x), \quad f''(x) = 6x + \cos(x), \quad f'''(x) = 6 - \sin(x), \quad f^{(4)}(x) = -\cos(x).$$

So, $|f^{(4)}(x)| = \cos(x)$ and since $-1 \le \cos(x) \le 1$ for any value x, we may choose $K = 1$. Then,

$$E_4 \le \frac{K(b-a)\Delta x^4}{180} = \frac{\pi^5}{180(8)}.$$

3. (10 marks) Evaluate the indefinite integral:

$$\int \cos(\ln x)\,dx.$$

Solution: First, we use substitution with $t = \ln x$ (equivalently, $x = e^t$), then $dt = \frac{dx}{x}$ since we have a composite function. So, $dx = x\,dt = e^t\,dt$, and

$$\int \cos(\ln x)\,dx = \int e^t \cos t\,dt.$$

Using integration by parts with $u = e^t$, $dv = \cos(t)\,dt$, $du = e^t\,dt$, and $v = \sin(t)$, we get:

$$\int e^t \cos t\,dt = e^t \sin(t) - \int e^t \sin(t)\,dt.$$

For the remaining integral, using integration by parts again with $u_1 = e^t$, $dv_1 = \sin(t)\,dt$, $du_1 = e^t\,dt$, and $v_1 = -\cos(t)$, we get:

$$\int e^t \sin t\,dt = -e^t \cos(t) + \int e^t \cos(t)\,dt.$$

So,

$$\int e^t \cos t\,dt = e^t \sin(t) - \int e^t \sin(t)\,dt = e^t \sin t + e^t \cos(t) - \int e^t \cos(t)\,dt$$

$$\Rightarrow 2\int e^t \cos t\,dt = e^t \sin(t) + e^t \cos(t) + C \Rightarrow \int e^t \cos(t)\,dt = \frac{e^t \sin(t) + e^t \cos(t)}{2} + C.$$

Replacing with $t = \ln x$ and simplifying we get:

$$\int \cos(\ln x)\,dx = \frac{x \sin(\ln x) + x \cos(\ln x)}{2} + C.$$

4. (10 marks) Solve the initial value problem:

$$\frac{dy}{dt} = \frac{ty^3}{\sqrt{1+t^2}}, \qquad y(0) = -1.$$

Express the answer in its explicit form.

Solution: We have:

$$\frac{dy}{dt} = \frac{ty^3}{\sqrt{1+t^2}} \Leftrightarrow \frac{dy}{y^3} = \frac{t}{\sqrt{1+t^2}}\,dt.$$

Next, we want to integrate each side with respect to the respective variables. The left hand side yields:

$$\int y^{-3}\,dy = -\frac{1}{2y^2} + C.$$

For the integral $\int \frac{t}{\sqrt{1+t^2}}\,dt$, we use substitution with $x = 1 + t^2$ and $dx = 2t\,dt$. Then,

$$\int \frac{t}{\sqrt{1+t^2}}\,dt = \int \frac{u^{-1/2}}{2}\,dx = u^{1/2} + C = \sqrt{1+t^2} + C.$$

So, $-\frac{1}{2y^2} = \sqrt{1+t^2} + C$. To find C, we use the initial condition $y(0) = -1$, and get:

$$-\frac{1}{2} = \sqrt{1+0} + C \Rightarrow C = -3/2.$$

So, the solution to the initial value problem in its implicit form is:

$$\begin{aligned} -\frac{1}{2y^2} &= \sqrt{1+t^2} - 3/2 \\ \Rightarrow 2y^2 &= \frac{1}{3/2 - \sqrt{1+t^2}} \\ y^2 &= \frac{1}{3 - 2\sqrt{1+t^2}} \\ y &= \pm\frac{1}{\sqrt{3-2\sqrt{1+t^2}}}. \end{aligned}$$

Since $y(0) = -1$, the explicit form of the solution is $y = -\frac{1}{\sqrt{3-2\sqrt{1+t^2}}}$.